Functional Approaches to Programming

Higher Order Functions

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Outline

Plan

Reminder: Higher Order

- **Christopher Strachey**
  - naming (variables)
  - aggregation (structures)
  - anonymous manipulation
  - function argument
  - function return value
  - dynamic construction
  - ...

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Definition, Naming, Assignment

- **Classical approach**

Lisp

```
(defun backwards (lst) (reverse lst))
```

Haskell

```
backwards :: [a] -> [a]
backwards xs = reverse xs
```

- **Higher order**

Lisp (Lisp-2 !)

```
;; From function to function:
(setf (symbol-function 'backwards) #'reverse)

;; From variable to variable:
(setq function2 function1)

;; From function to variable:
(setq function2 #'function1)

;; From variable to function:
(setf (symbol-function 'function2) function1)
```

Haskell

```
backwards :: [a] -> [a]
backwards = reverse
```

Scheme (Lisp-1)

```
(define backwards reverse)
(set! function2 function1)
```

- **Aggregation** (`struct`, `data`, *etc.*)

Local Variables / Functions

- **Local variables**

Lisp

```
(defun f (x)
  (let ((a (* x x))
        (b (+ (* x x) 1)))
    (+ (/ a b) (/ b a))))
```

```
(defun f (x)
  (let* ((a (* x x))
         (b (+ a 1)))
    (+ (/ a b) (/ b a))))
```

Haskell

```
f :: Float -> Float
f x = let a = x * x
          b = a + 1
      in a / b + b / a
```

```
f :: Float -> Float
f x = a / b + b / a
  where a = x * x
        b = a + 1
```

⚠ **Lisp:** no mutual references in `let` (use `let*`)

Local Variables / Functions

- **Local functions**
  - **Haskell:** `let` and `where`
  - **Lisp:** `flet` / `labels` (Lisp-2)

Lisp

```
(defun ssq (x y)
  (flet ((square (x) (* x x)))
    (+ (square x) (square y))))
```

Haskell

```
ssq :: Float -> Float -> Float
ssq x y = let square x = x * x
  in square x + square y
```

```
ssq :: Float -> Float -> Float
ssq x y = square x + square y
  where square x = x * x
```

Partial Application / Operator Cut (Haskell)

Haskell

```haskell
multiply :: Float -> Float -> Float
multiply a b = a * b

-- multiply a b
```

- `->` is right-associative
  `Float -> (Float -> Float)`
- Application is left-associative:
  `multiply 2 5` ⇔ `(multiply 2) 5`
- **Curryfication:** Haskell functions are unary
  - **Partial application:**
    `multiply 2 :: Float -> Float`
  - **Operator cut:**
    `(+2) (>3) (3:) ("error: "++)` *etc.*

Application: `nrsqrt` II, the Return

Haskell

```haskell
nrsqrt :: Float -> Float -> Float
-- nrsqrt x delta = nrfind 1.0 x delta
nrsqrt = nrfind 1.0

nrfind :: Float -> Float -> Float -> Float
nrfind yn x delta
  | nrfound yn x delta = yn
  | otherwise = nrfind (nrnext yn x) x delta

nrfound :: Float -> Float -> Float -> Bool
nrfound yn x delta = abs (x - yn * yn) <= delta

nrnext :: Float -> Float -> Float
nrnext yn x = (yn + x / yn) / 2
```

Imperative Pseudo Higher Order

C

```c
typedef ... list;

list reverse (list l)
{
  /* ... */
}
/*
   new_lst = reverse (lst);
*/

typedef list (* list_to_list_f) (list);
list_to_list_f backwards = reverse;
/*
   new_lst = (* backwards) (lst);
   new_lst = backwards (lst);
*/
```


Plan

Anonymous Functions (“Lambda”)

- **Principle**
  - Possibility to *not* name a function
  - Litteral use
- **Denotation**

Lisp

```
(lambda (x) (* 2 x))
```

Haskell

```
\x -> 2 * x
```

- **Application**

```
((lambda (x) (* 2 x)) 4)
```

```
(\x -> 2 * x) 4
```

- **Remark:** macro `lambda` (Lisp)

```
(lambda (x y ...) ...)
=> (function (lambda (x y ...) ...))
=> #'(lambda (x y ...) ...)
```

Application to Local Contexts

- **Conceptual equivalence**

Lisp

```
(defun f (x)
  (let ((a (* x x))
        (b (+ (* x x) 1)))
    (+ (/ a b) (/ b a))))
```

```
(defun f (x)
  ((lambda (a b) (+ (/ a b) (/ b a)))
   (* x x) (+ (* x x) 1)))
```

Haskell

```
f :: Float -> Float
f x = let a = x * x
          b = a + 1
      in a / b + b / a
```

```
f :: Float -> Float
f x = (\a b -> a / b + b / a)
      (x*x) (x*x+1)
```

⚠ **Remark:** no mutual references possible

Application to Local Contexts

- Imbrication

Lisp

```
(defun f (x)
  (let* ((a (* x x))
         (b (+ a 1)))
    (+ (/ a b) (/ b a))))
```

```
(defun f (x)
  (let ((a (* x x)))
    (let ((b (+ a 1)))
      (+ (/ a b) (/ b a)))))
```

```
(defun f (x)
  ((lambda (a)
     ((lambda (b)
        (+ (/ a b) (/ b a)))
      (+ a 1)))
   (* x x)))
```

Haskell

```
f :: Float -> Float
f x = let a = x * x
          b = a + 1
      in a / b + b / a
```

```
f :: Float -> Float
f x = let a = x * x
      in let b = a + 1
         in a / b + b / a
```

```
f :: Float -> Float
f x = (\a  ->
          (\b ->
              a / b + b / a)
          (a+1))
      (x*x)
```


Imperative Pseudo-Anonymity

- Explicit Blocks

C

```c
if (expression)
{
  /* Pseudo-anonymous procedure ... */
}
else
{
  /* Pseudo-anonymous procedure ... */
}
```

Plan

Functional Application

- **Reminder:** application / β-reduction (λ-calculus)
- `apply` and `funcall`

Lisp

```
(+ 1 2 3 4 5)
(apply #'+ '(1 2 3 4 5))
(apply #'+ 1 2 3 '(4 5))
(funcall #'+ 1 2 3 4 5)
```

(Missing) Abstraction

$$\sum_{n=a}^{b} f(n) = f(a) + \ldots + f(b)$$

Lisp

```
(defun sint (a b)
  (if (> a b)
    0
    (+ a (sint (1+ a) b))))
```

```
(defun ssq (a b)
  (if (> a b)
    0
    (+ (* a a) (ssq (1+ a) b))))
```

```
(defun spi (a b)
  (if (> a b)
    0
    (+ (/ 1.0 (* a (+ a 2)))
       (spi (+ a 4) b))))
```

Haskell

```
sint :: Int -> Int -> Int
sint a b
  | a > b = 0
  | otherwise = a + sint (a+1) b
```

```
ssq :: Int -> Int -> Int
ssq a b
  | a > b = 0
  | otherwise = a*a + ssq (a+1) b
```

```
spi :: Int -> Int -> Float
spi a b
  | a > b = 0
  | otherwise = 1.0 / fromIntegral (a * (a+2))
              + spi (a+4) b
```

(Explicit) Abstraction

Lisp

```
(defun sigma (a b term next)
  (if (> a b)
    0
    (+ (funcall term a)
       (sigma (funcall next a) b term next))))
```

```
(defun sint (a b)
  (funcall #'sigma a b #'identity #'1+))

(defun ssq (a b)
  (funcall #'sigma a b (lambda (x) (* x x)) #'1+))

(defun spi-term (a)
  (/ 1.0 (* a (+ a 2))))
(defun spi-next (a)
  (+ a 4))
(defun spi (a b)
  (funcall #'sigma a b #'spi-term #'spi-next))
```

Haskell

```
sigma :: Num a => Int -> Int
  -> (Int -> a) -> (Int -> Int) -> a
sigma a b term next
  | a > b = 0
  | otherwise = term a + sigma (next a) b term next
```

```
sint :: Int -> Int -> Int
sint a b = sigma a b id (+1)

ssq :: Int -> Int -> Int
ssq a b = sigma a b (\x -> x*x) (+1)

spiterm :: Int -> Float
spiterm a = 1.0 / fromIntegral (a * (a+2))

spinext :: Int -> Int
spinext a = a + 4

spi :: Int -> Int -> Float
spi a b = sigma a b spiterm spinext
```

Application: `nrsqrt` III, the Revenge

$$\delta \text{ approximation of } \lim_{n \to +\infty} u_n(x)$$

Lisp

```
(defun limit (found next x delta)
  (limit1 1.0 found next x delta))

(defun limit1 (yn found next x delta)
  (if (funcall found yn x delta)
    yn
    (limit1 (funcall next yn x) found next x delta)))


(defun nrfound (yn x delta)
  (<= (abs (- x (* yn yn))) delta))

(defun nrnext (yn x)
  (/ (+ yn (/ x yn)) 2))

(defun nrsqrt (x delta)
  (limit #'nrfound #'nrnext x delta))
```


Application: `nrsqrt` III, the Revenge

$$\delta \text{ approximation of } \lim_{n \to +\infty} u_n(x)$$

Haskell

```haskell
limit :: (Float -> Float -> Float -> Bool) -> (Float -> Float -> Float) -> Float -> Float -> Float
-- limit found next x delta = limit1 1.0 found next x delta
limit = limit1 1.0

limit1 :: Float -> (Float -> Float -> Float -> Bool) -> (Float -> Float -> Float) -> Float -> Float
  -> Float
limit1 yn found next x delta
  | found yn x delta = yn
  | otherwise = limit1 (next yn x) found next x delta


nrfound :: Float -> Float -> Float -> Bool
nrfound yn x delta = abs (x - yn * yn) <= delta

nrnext :: Float -> Float -> Float
nrnext yn x = (yn + x / yn) / 2

nrsqrt :: Float -> Float -> Float
-- nrsqrt x delta = limit nrfound nrnext x delta
nrsqrt = limit nrfound nrnext
```

Pattern 1: Mapping

- **Principle:** process all elements in a list
- **Example**

Lisp

```
(defun dbl (l)
  (if (null l)
    nil
    (cons (* 2 (first l)) (dbl (rest l)))))
```

Haskell

```
dbl :: [Int] -> [Int]
dbl [] = []
dbl (x:xs) = 2*x : dbl xs
```

- **Lisp:** `(mapcar func list &rest lists)`
- **Haskell:** `map :: (a -> b) -> [a] -> [b]`
- **Application**

```
(defun dbl (l)
  (mapcar (lambda (x) (* 2 x)) l))
```

```
dbl :: [Int] -> [Int]
dbl = map (*2)
```

Pattern 1bis: Generalized Mapping

- **Principe:** mapping on several lists
- **Example**

Lisp

```
(defun list+ (l1 l2)
  (if (or (null l1) (null l2))
    nil
    (cons (+ (first l1) (first l2))
          (list+ (rest l1) (rest l2)))))
```

Haskell

```
(!+) :: [Int] -> [Int] -> [Int]
(!+) (x:xs) (y:ys) = (x + y) : xs !+ ys
(!+) _      _      = []
```

- **Lisp:** variadic `mapcar`
- **Haskell:** `zipWith :: (a -> b -> c) -> [a] -> [b]-> [c]`
- **Application**

```
(defun list+ (l1 l2) (mapcar #'+ l1 l2))
```

```
(!+) :: [Int] -> [Int] -> [Int]
(!+) = zipWith (+)
```


Pattern 2: Filtering

- **Principle:** retain or reject elements from a list
- **Example**

Lisp

```
(defun pst (l)
  (cond ((null l) nil)
        ((> (first l) 0)
         (cons (first l) (pst (rest l))))
        (t (pst (rest l)))))
```

Haskell

```
pst :: [Int] -> [Int]
pst [] = []
pst (x:xs)
  | x > 0 = x : pst xs
  | otherwise = pst xs
```

- **Lisp:** `(remove-if[-not] pred list &key ...)`
- **Haskell:** `filter :: (a -> Bool) -> [a] -> [a]`
- **Application**

```
(defun pst (l)
  (remove-if (lambda (x) (< x 0)) l))
```

```
pst :: [Int] -> [Int]
pst = filter (>0)
```

Pattern 3: Folding / Reduction

- **Principle:** combine all elements from a list
- **Example**

Lisp

```
(defun listsum (l)
  (cond ((null l) (error "empty list"))
        ((null (rest l)) (first l))
        (t (+ (first l) (listsum (rest l))))))
```

Haskell

```
listsum :: [Int] -> Int
listsum [x] = x
listsum (x:xs) = x + listsum xs
```

- **Lisp:** `(reduce func seq &key ...)`
- **Haskell:** `foldr1 :: (a -> a -> a) -> [a] -> a`
- **Application**

```
(defun listsum (l) (reduce #'+ l))
```

```
listsum :: [Int] -> Int
listsum = foldr1 (+)
```


Pattern 3bis: Generalized Folding

- **Principle:** folding with an initial value (case of the empty list)
- **Lisp:** `:initial-value` key to `reduce`
- **Haskell:** `foldr`

```haskell
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f s []     = s
foldr f s (x:xs) = f x (foldr f s xs)
```

- **Application**

Lisp

```lisp
;; Nothing to do! (+) => 0
(defun listsum (l) (reduce #'+ l))
```

Haskell

```haskell
listsum :: [Int] -> Int
listsum = foldr (+) 0
-- sum !
```

ssq II, the Return

- **Reminder:** recursive version

Lisp

```
(defun ssq (n)
  (if (= n 1)
    1
    (+ (* n n) (ssq (1- n)))))
```

Haskell

```
ssq :: Int -> Int
ssq 1 = 1
ssq n = n*n + ssq (n-1)
```

- **New version:** “foldmap” on the list of integers

```
(defun ssq (n)
  (reduce #'+
      (mapcar (lambda (x) (* x x))
        (intlist n))))
```

```
ssq :: Int -> Int
ssq n = foldr1 (+) (map (\x -> x * x) [1..n])
-- ssq n = sum (map (\x -> x * x) [1..n])
```

List "Comprehensions" (Haskell)

$$\{f(x)|x \in E, C_1(x), C_2(x), \ldots\}$$

- **Literally:** "the set of all $f(x)$ such that $x \in E$ and all properties $C_i(x)$ are satisfied"
- **Generator:** `[ f | e <- l, c1, c2, ...]`
- **Examples**

```
[ 2*n    | n <- [2,3,4] ] -- [4,6,8]
[ n == 3 | n <- [2,3,4] ] -- [False, True, False]
[ 2*n    | n <- [2,3,4], n == 3] -- [6]
```

- **With pattern matching**

```
[ m*n | [m,n] <- [[2,3],[4,5],[6,7]], m > 2] -- [20, 42]
```

Remarks on Comprehensions

- Used as filters

```
import Char

digits :: String -> String
digits str = [ c | c <- str, isDigit c ]
```

- Used within a function's body

```
allEven :: [Int] -> Bool
allEven xs = (xs == [e | e <- xs, isEven e])

allOdd :: [Int] -> Bool
allOdd xs = ([] == [e | e <- xs, isEven e])
```

⚠ **Caution:** variables are comprehension-local

```
bogusFind :: Int -> [[Int]] -> [[Int]]
bogusFind x ys = ([ x:zs | x:zs <- ys])
```

Plan

Generalities

- **Obvious principle:** `#'+`, `func`, *etc.*
- **Hidden functional returns**
  - Partial application: `(multiply 2)`
  - Operator cut: `(+3)`
- **Returning existing functions**

Lisp

```
(defun dispatch (obj)
  (typecase obj
    (class1 #'method1)
    (class2 #'method2)
    ...))
```

Lisp

```
(defun +/- (plusp)
  (if plusp #'+ #'-))
;; (funcall (+/- t) 4 2)
```

Haskell

```
plusOrMinus :: Bool -> (Int -> Int -> Int)
plusOrMinus True = (+)
plusOrMinus False = (-)
-- (plusOrMinus t) 4 2
```

C

```
typedef int (* int_f) (int, int);
int_f plus, minus;

int_f plus_or_minus (int pls)
{
  return pls ? plus : minus;
}
/* (* plus_or_minus (1)) (4, 2); */
```

Mathematical Composition ($f \circ g$)

- **Example:** logical complement

Lisp

```
(defun isodd (n) (not (evenp n))) ;; oddp
```

```
(setf (symbol-function 'isodd) ;; oddp
      (complement #'evenp))
```

Haskell

```
isOdd :: Integer -> Bool -- odd
isOdd n = not (even n)
```

```
isOdd :: Integer -> Bool -- odd
isOdd = not . even
```

- **Advantage:** orthogonality
  Obsolescence of the -not forms (Lisp)

```
(remove-if-not #'isodd lst)
(remove-if (lambda (x) (not (evenp x))) lst)
(remove-if (complement #'evenp) lst)
```

- **Operator** . (Haskell)
  `(.) :: (b -> c) -> (a -> b) -> (a -> c)`

Iterative Composition (f^n)

- Recursion

Haskell

```haskell
iter :: Int -> (a -> a) -> (a -> a)
iter n f
    | n > 0 = f . iter (n - 1) f
    | otherwise = id
```

- Reduction

```haskell
iter :: Int -> (a -> a) -> (a -> a)
iter n f = foldr (.) id (replicate n f)
```

Anonymous Returns

- Logical complement

Lisp

```lisp
(defun complement (f)
  (lambda (&rest args) (not (apply f args))))
```

- Operator cut / partial application

```lisp
(defun adder (n) (lambda (x) (+ x n)))
;; (funcall (adder 3) 1) => 4
```

Haskell

```haskell
adder :: Int -> (Int -> Int)
adder n = \x -> x + n
-- (adder 3) 1 => 4
```

- ssq III, the Revenge

```lisp
(defun foldmap (f m)
  (lambda (x) (reduce f (mapcar m x))))

(defun ssq (n)
  (funcall (foldmap #'+ (lambda (x) (* x x)))
    (intlist n)))
```

```haskell
foldmap :: (c -> c -> c) -> (c -> c)
  -> ([c] -> c)
foldmap f m = (foldr1 f) . (map m)

ssq :: Int -> Int
ssq n = foldmap (+) (\x -> x*x) [1..n]
```


Plan

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Concept of “Data”

- **Imperative view** (historical)
  - Representation + Manipulation
  - Aggregates + Functions
  - Classes + Methods
  - *etc.*
- **Data abstraction**
  - Software Engineering Principle
  - Distinguish interface from implementation
  - Any implementation can do
- **Example**

C

```c
complex_t make_complex (float real, float img);
float real_part (complex_t complex);
float img_part (complex_t complex);
```

C++

```cpp
new Complex (float real, float img);
float complex.real_part ();
float complex.img_part ();
```

🤔 ((purely) functional) Implémentation ?

Concept of “Data”

Lisp

```
(defun make-complex (real img)
  (lambda (selector)
    (case selector
      (:real real)
      (:img img))))

(defun real-part (complex)
  (funcall complex :real))

(defun img-part  (complex)
  (funcall complex :img))
```

Haskell

```
data Selector = Real | Img
type Complex = Selector -> Float

makeComplex :: Float -> Float -> Complex
makeComplex real img = select
  where select Real = real
        select Img = img

realPart :: Complex -> Float
realPart complex = complex Real

imgPart :: Complex -> Float
imgPart complex = complex Img
```

- No data structure / aggregate (object = function)
- Message passing strategy, *cf.* AOP!